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# Steady-State Response of Vibrating Systems to Periodic Pulse Excitation


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## Introduction

MACHINES and structures are often subjected to periodic pulse excitations. Examples include punch presses, rolling mills, gear teeth, and internal combustion engines. Two common methods to obtain the steady-state response of such a system are:

1) The Fourier series method, in which the periodic pulse excitation is expanded in a Fourier series and the system response to each term in the series is summed.

2) The Laplace transform method.

Both methods involve summation of series. In this Note, a simple closed-form expression is obtained for the steady-state response of a system, using the convolution integral along with the unit impulse response.[1]

## Analysis

The equation of motion of a simple vibrating system with mass $m$, stiffness $k$, and damping coefficient $c$ can be written as

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=F(t)/m$$

$$\begin{aligned} F(t)&=F_0 \qquad nT<t<nT+t_f \\ &=0 \qquad nT+t_f<t<(n+1)T \qquad n=0,1,2,\ldots \\ &x=\dot{x}=0 \text{ for } t<0 \end{aligned} \tag{1}$$

where $\omega_n=(k/m)^{1/2}$ is the system natural frequency, $\zeta=c/2m\omega_n$ the damping ratio, $T$ the period of the excitation function, and $t_f$ the duration of the pulse within a period. The periodic pulse excitation $F(t)$ is shown in Fig. 1.

Assuming that steady-state conditions are obtained after sufficiently large $n=N$, the system response will satisfy the following conditions:

$$x(NT)=x[(N+1)T]=A$$

$$\dot{x}(NT)=\dot{x}[(N+1)T]=B \tag{2}$$

where $A$ and $B$ are to be determined.

The system response during $NT<t<(N+1)T$ may be written as

$$x(\xi)=\exp(-\zeta\omega_n\xi)\left[A\cos\omega_d\xi+\left(\frac{B+\zeta\omega_nA}{\omega_d}\right)\sin\omega_d\xi\right]+\int_0^\xi h(\xi-\tau)F(\tau)\,\mathrm{d}\tau \tag{3}$$

where $\xi=t-NT$ and $h(t)$ is unit impulse response of the system. Substituting for $h(t)$ (Ref. 1) and $F(t)$ in Eq. (3), the following response is obtained:

$$\begin{aligned} x(\xi)&=\exp(-\zeta\omega_n\xi)\left[A\cos\omega_d\xi+\left(\frac{B+\zeta\omega_nA}{\omega_d}\right)\sin\omega_d\xi\right] \\ &\quad+\frac{1}{m\omega_d\omega_n^2}\{\omega_d-\exp(-\zeta\omega_n\xi)[\zeta\omega_n\sin\omega_d\xi+\omega_d\cos\omega_d\xi]\} \qquad 0<\xi<t_f \\ &=\exp(-\zeta\omega_n\xi)\left[A\cos\omega_d\xi+\left(\frac{B+\zeta\omega_nA}{\omega_d}\right)\sin\omega_d\xi\right] \\ &\quad+\frac{1}{m\omega_d\omega_n^2}\{\exp(-\zeta\omega_n\xi^*)[\zeta\omega_n\sin\omega_d\xi^*+\omega_d\cos\omega_d\xi^*] \\ &\quad-\exp(-\zeta\omega_n\xi)[\zeta\omega_n\sin\omega_d\xi+\omega_d\cos\omega_d\xi]\} \qquad t_f<\xi<T \end{aligned} \tag{4}$$

where $\xi^*=\xi-t_f$. Utilizing the conditions of Eq. (2) in Eq. (4), it is possible to obtain equations of the form:

$$a_{11}A+a_{12}B=a_{13} \qquad a_{21}A+a_{22}B=a_{23} \tag{5}$$

The constants $A$ and $B$ are obtained by solving Eq. (5) as

$$A=(a_{13}a_{22}-a_{23}a_{12})/(a_{11}a_{22}-a_{21}a_{12})$$

$$B=(a_{11}a_{23}-a_{21}a_{13})/(a_{11}a_{22}-a_{21}a_{12}) \tag{6}$$

The steady-state response is obtained by using these values of $A$ and $B$ in Eq. (4).

The response of the system obtained by summing the responses of the system to each term in the Fourier series

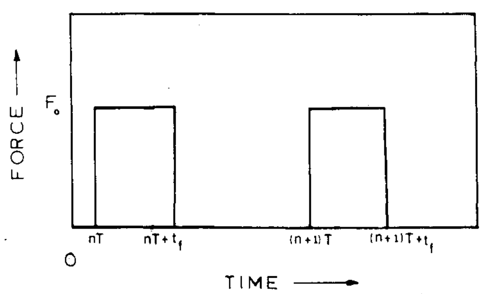


Fig. 1 Periodic pulse excitation.

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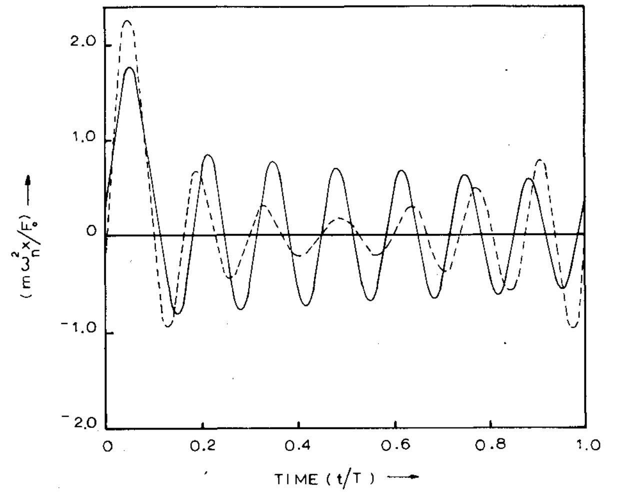


**Fig. 2 System response in a period of excitation (— convolution integral, ---- Fourier analysis, $\zeta = 0.01$, $t_f/T = 0.1$, $\eta = 0.134$).**

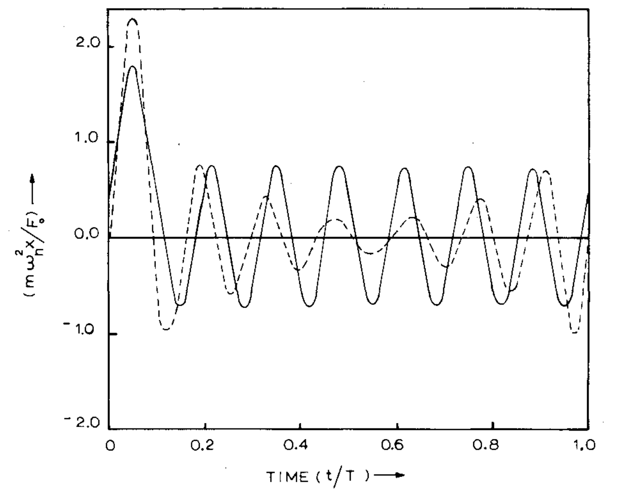


**Fig. 3 System response in a period of excitation (— convolution integral, ---- Fourier analysis, $\zeta = 0.0$, $t_f/T = 0.1$, $\eta = 0.134$).**

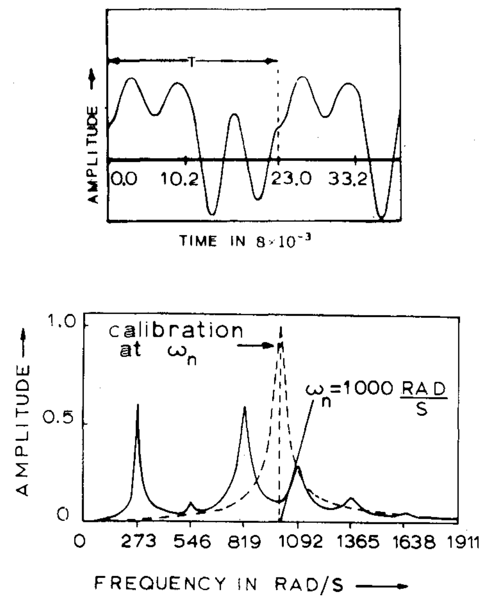


**Fig. 4 System response in time and frequency domains from analog computer model.**

expansion of the force $F(t)$ is given by

$$x(t) = \frac{F_0}{m\omega_n^2} \sum_{i=1}^{N} H_i(\omega) \left[ a_i \cos\left(\frac{2\pi i t}{T} - \phi_i\right) + b_i \sin\left(\frac{2\pi i t}{T} - \phi_i\right)\right] + \frac{F_0 a_0}{m\omega_n^2} \tag{7}$$

where

$$H_i(\omega) = 1/[(1-\eta_i^2)^2 + 4\zeta^2\eta_i^2]^{1/2}$$

$$\eta_i = 2\pi i/\omega_n T, \quad \phi_i = \tan^{-1}[2\zeta\eta_i/(1-\eta_i^2)]$$

$$a_0 = t_f/T, \quad a_i = (F_0/i\pi)\sin i\omega t_f$$

$$b_i = (F_0/i\pi)(i - \cos i\omega t_f), \quad \omega = 2\pi/T \tag{8}$$

Using Laplace transform techniques,[2] the $x(t)$ response may be written as

$$x(t) = L^{-1}[X(s)] = x_1(t) + x_2(t) \tag{9}$$

where

$$x_1(t) = \exp(-\zeta\omega_n t)\left\{x(0)\cos\omega_d t + \left[\frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}\right]\sin\omega_d t\right\} \tag{10}$$

$$x_2(t) = f(t) - f(t-t_f)u_{t_f}(t) + f(t-T)u_T(t) + \ldots + f[t-nT-t_f]u_{nT+t_f}(t) + f(t-nT)u_{nT}(t) + \ldots \tag{11}$$

$$f(t) = \frac{1}{m\omega_n^2}\left[1 - \exp(-\zeta\omega_n t)\left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right)\right] \tag{12}$$

The steady-state response of the system can be obtained by considering sufficiently large numbers of terms in Eq. (11).

## Discussion

Steady-state responses during one period of the excitation obtained from Eq. (4) using the convolution integral method and from Eq. (7) using the Fourier series method are shown in Figs. 2 and 3, respectively, for damped and undamped systems.

The responses using Fourier analysis, shown in Figs. 2 and 3, were computed considering 74 terms in the Fourier series expansion of the periodic pulse excitation. The highest harmonic considered is 7.4 times the natural frequency of the system and yet this response does not converge to the exact response obtained by the convolution integral method.

In a loose sense it can be said that, within one period, Eq. (4) giving the response using convolution integral and Eq. (11) giving response using the Laplace transform, contain "decaying oscillations at the resonance frequency of the system," if the system is damped and that the oscillations are at the natural frequency $\omega_n$, if the system is undamped as seen in Figs. 2 and 3, respectively. However, a Fourier expansion of the periodic response process will contain only the harmonics of this periodic process and not the system resonance frequency. The response will contain the system resonance frequency only if it coincides with one of the harmonics of the periodic excitation. For an undamped system, the steady-state response within a period will contain oscillations at the system natural frequency in the range $(NT+t_f) < t < (N+1)T$, however, for $NT < t < (NT+t_f)$ the response is different and, hence, a Fourier analysis of the periodic process will not show any harmonic content at the system natural frequency. This phenomenon is verified by the following experiment on a system modeled on an analog computer.

The damped vibrating system had $\omega_n = 1000$ rad/s and $\zeta = 0.05$. The period of the pulse excitation was 0.023 s and the pulse width was $(t_f/T) = 0.444$. The time domain response and the response spectra are shown in Fig. 4. The response spectra in Fig. 4b contain only the frequencies contained in the excitation function.

### Acknowledgments

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# Static Analysis of Stiffened Plates

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### Nomenclature

| | |
|---|---|
| $A_x, A_y$ | = area of stiffener |
| $a, b$ | = dimensions of plate |
| $D$ | $= Eh^3/12(1-\mu^2)$ |
| $E$ | = Young's modulus |
| $h$ | = plate thickness |
| $I$ | $= h^3/12$ |
| $I_x, I_y$ | = moment of inertia of the stiffener about the midplane of the plate |
| $q$ | = uniform lateral load |
| $S_x, S_y$ | = moment of area of the stiffener about the midplane of the plate |
| $u, v, w$ | = in-plane and lateral displacements of midplane of the plate |

### Introduction

In many of the available references on the analysis of stiffened plates, the approximate method proposed by Huber is used. Based on the orthotropic plate theory, this method analyzes the plate stiffener system as a plate of equivalent uniform thickness. It neglects the in-plane displacement of the middle plane of the plate.

In an improved method presented by Clifton et al.,[1] the eccentricity and torsional rigidity of the stiffeners are taken into account, the effect of the stiffeners is smeared out. The governing equations are solved for a simply supported plate and a plate with bridge-type boundary conditions.

In this Note, stiffened clamped rectangular plates have been analyzed taking the eccentricity of the stiffener into consideration. There is no available solution for this problem. The governing equations in terms of the in-plane and lateral displacements of the midplane of the plate are solved using a series consisting of beam functions and Galerkin's procedure. Numerical work has been done for several aspect ratios. The results for the deflection and stresses are presented graphically.

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### Differential Equations

The governing differential equations for the eccentrically stiffened plate (Fig. 1) can be written in nondimensional form[1] as

$$a_1 U'' + a_2 V^{\circ\prime} + a_3 U^{\circ\circ} + a_4 W''' = 0 \tag{1}$$

$$b_1 V^{\circ\circ} + b_2 U^{\circ\prime} + b_3 V'' + b_4 W^{\circ\circ\circ} = 0 \tag{2}$$

$$c_1 W'''' + c_2 W''^{\circ\circ} + c_3 W^{\circ\circ\circ\circ} + c_4 U''' + c_5 V^{\circ\circ\circ} = qa^4/Dh \tag{3}$$

where $\bar{x} = x/a$, $\bar{y} = y/b$, $U = ua/h^2$, $V = vb/h^2$, and $W = w/h$. The coefficients are defined in the Appendix. ( )′ and ( )° denote differentiation with respect to $\bar{x}$ and $\bar{y}$.

The boundary conditions are

$$\text{At } \bar{x} = 0 \text{ and } 1 \qquad W = W' = U = V = 0 \tag{4a}$$

$$\text{At } \bar{y} = 0 \text{ and } 1 \qquad W = W^{\circ} = U = V = 0 \tag{4b}$$

### Solution

The solution that satisfies all of the boundary conditions of Eqs. (4) is assumed to be in the form

$$U = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} x'_m y_n \tag{5a}$$

$$V = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} x_m y_n^{\circ} \tag{5b}$$

$$W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} x_m y_n \tag{5c}$$

where $x_m$ and $y_n$ are the beam functions corresponding to the clamped-clamped edge conditions, i.e.,

$$x_m = (\cosh\beta_m \bar{x} - \cos\beta_m \bar{x}) - \alpha_m (\sinh\beta_m \bar{x} - \sin\beta_m \bar{x})$$

$$y_n = (\cosh\beta_n \bar{y} - \cos\beta_n \bar{y}) - \alpha_n (\sinh\beta_n \bar{y} - \sin\beta_n \bar{y})$$

Substituting Eqs. (5) in Eqs. (1-3) and using Galerkin's method, the solution is obtained. The details regarding the values of $\alpha_m$ and $\beta_m$ and the evaluation of the integrals can be found in Refs. 2 and 3.

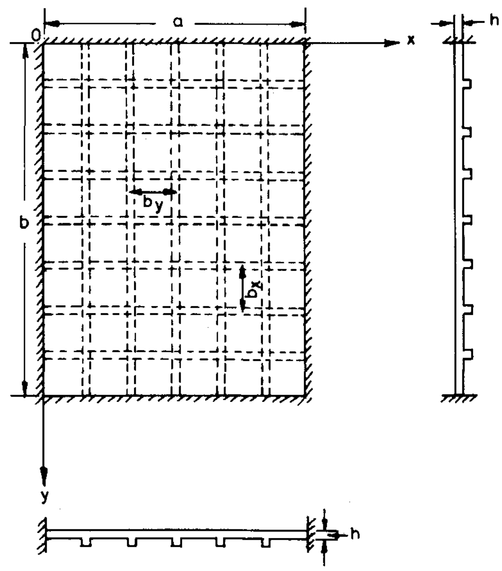


**Fig. 1 Plate with eccentric stiffeners.**